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HUYGEN'S PRINCIPLE IN A MOVING ISOTROPIC HOMOGENEOUS AND LINEAR MEDIUM+

Chen-To Tai
Radiation Laboratory
The University of Michigan
Ann Arbor, Michigan, USA

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ABSTRACT

This work contains an analytical description of Huygen's Principle for an electromagnetic field in a moving isotropic, homogeneous and linear medium. By starting with Maxwell-Minkowski equations it is possible to construct a combined field equation in the manner of Bateman and Itoh. The resultant equation is then integrated with the aid of an appropriate dyadic Green's function which can be found by means of the operational method originally due to Levine and Schwinger.

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Introduction

The electrodynamics of moving media has attracted much interest recently as a result of the work by Professor L. V. Boffi[1] of the Instituto Tecnologico de Aeronautica, and by Professor L. J Chu[2] of Massachusetts Institute of Technology. The relations between these new formulations and Minkowski's classical work were reviewed by this author in a recent article[3].

In this paper, we shall present an analytical description of Huygens' Principle for an electromagnetic field in a moving isotropic, homogeneous, and linear medium. The treatment follows very closely the technique developed previously for a stationary medium[4]. A combined field equation is first constructed out of the Maxwell-Minkowski equations. It is then integrated with the aid of an appropriate dyadic Green's function[5].

Maxwell-Minkowski Equations

The Maxwell's equations for moving media have the same form as for stationary media. For harmonically oscillative fields with a time convention $e^{j\omega t}$, they are

$$\nabla \times \bar{E} = -j\omega \bar{B} \tag{1}$$

$$\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} \tag{2}$$

The relations between the field vectors for a moving isotropic medium were found by Minkowski[6] based upon the special theory of relativity. They are

$$\bar{D} + \frac{1}{c^2}\bar{v} \times \bar{H} = \epsilon(\bar{E} + \bar{v} \times \bar{B}) \tag{3}$$

[redacted]

$$\bar{B} - \frac{1}{c^2}\bar{v} \times \bar{E} = \mu(\bar{H} - \bar{v} \times \bar{D}) \tag{4}$$

where ϵ and μ denote, respectively, the permittivity and permeability of the medium at rest, which is assumed to be loseless. $\bar{v}$ and c denote, respectively, the velocity of moving medium and the speed of light in vacuum. In the subsequent analysis, we assume

$$\bar{v} = v\hat{z}. \tag{5}$$

The above condition is not much of a restriction since a coordinate transformation of the result can easily take care of the general case.

By solving $\bar{D}$ and $\bar{B}$ from (3-4) in terms of $\bar{E}$ and $\bar{H}$ with $\bar{v}$ given by (5) we obtain the following relations:

$$\bar{D} = \epsilon\,\bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H} \tag{6}$$

$$\bar{B} = \mu\,\bar{\bar{\alpha}} \cdot \bar{H} - \bar{\Omega} \times \bar{E} \tag{7}$$

where

$$\bar{\Omega} = \frac{(n^2 - 1)\beta}{(1 - n^2\beta^2)c}\hat{z} \tag{8}$$

$$\beta = v/c \tag{9}$$

$$n = \left(\frac{\mu\,\epsilon}{\mu_o\,\epsilon_o}\right)^{\frac{1}{2}} \tag{10}$$

$$\bar{\bar{\alpha}} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a = \frac{1 - \beta^2}{1 - n^2\beta^2} \tag{11}$$

Substitution of (6-7) into (12) yields the Maxwell-Minkowski equations for a moving isotropic medium. They are

$$(\nabla - j\omega\bar{\Omega}) \times \bar{E} = -j\omega\mu\,\bar{\bar{\alpha}}\cdot\bar{H} \tag{12}$$

$$(\nabla - j\omega\bar{\Omega}) \times \bar{H} = \bar{J} + j\omega\epsilon\,\bar{\bar{\alpha}}\cdot\bar{E} \tag{13}$$

Integration of the Maxwell-Minkowski Equations

Equations (12-13) can be simplified by introducing two new field vectors $\bar{E}_1$ and $\bar{H}_1$ defined by

$$\bar{E} = \bar{E}_1\, e^{j\omega\Omega z} \tag{14}$$

$$\bar{H} = \bar{H}_1\, e^{j\omega\Omega z} \tag{15}$$

then

$$(\nabla - j\omega\bar{\Omega}) \times \bar{E} = e^{j\omega\Omega z}\, \nabla \times \bar{E}_1 \tag{16}$$

$$(\nabla - j\omega\bar{\Omega}) \times \bar{H} = e^{j\omega\Omega z}\, \nabla \times \bar{H}_1 \tag{17}$$

Equations (12-13), therefore, reduce to

$$\nabla \times \bar{E}_1 = -j\omega\mu\,\bar{\bar{\alpha}}\cdot\bar{H}_1 \tag{18}$$

$$\nabla \times \bar{H}_1 = \bar{J}\, e^{-j\omega\Omega z} + j\omega\epsilon\,\bar{\bar{\alpha}}\cdot\bar{E}_1 \tag{19}$$

To integrate (18-19), we shall adopt the technique[4] previously developed for stationary media. Following Bateman[7] and Itoh[8], we define a combined field vector

$$\bar{F}_1 = \bar{E}_1 + \Gamma \bar{H}_1 \tag{20}$$

where

$$\Gamma^2 = -\mu/\epsilon$$

or

$$\Gamma = \pm j(\mu/\epsilon)^{\frac{1}{2}} = jh\eta \tag{21}$$

The sign "± 1", denoted by h, is used as a separation operator similar to $\sqrt{-1}$ or j in complex number theory. We adopt the rule the $h^2 = 1$. By multiplying (19) by Γ and adding it to (18) one finds

$$\nabla \times \bar{F}_1 = K \bar{\bar{\alpha}} \cdot \bar{F}_1 + \Gamma \bar{J} e^{-j\omega\Omega z} \tag{22}$$

where

$$K = j\omega\epsilon\Gamma = -hk; \quad k = \omega(\mu\epsilon)^{\frac{1}{2}}. \tag{23}$$

Multiplying (22) by $\bar{\bar{\alpha}}^{-1}$, the reciprol of $\bar{\bar{\alpha}}$, and taking the curl of the resultant equation yields

$$\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \nabla \times \bar{F}_1) = K \nabla \times \bar{F}_1 + \Gamma \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{J} e^{-j\omega\Omega z}) \tag{24}$$

or

$$\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \nabla \times \bar{F}_1) - k^2 \bar{\bar{\alpha}} \cdot \bar{F}_1 = \bar{f} \tag{25}$$

where

$$\bar{f} = j\omega\mu \bar{J} e^{-j\omega\Omega z} + \Gamma \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{J} e^{-j\omega\Omega z}) \tag{26}$$

Equation (25) is the wave equation for the combined field vector $\bar{F}_1$. To integrate that equation, we introduce the dyadic Green's function $\bar{\bar{g}}$ which satisfies the differential equation:

$$\nabla \times \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}}) \right] - k^2 \bar{\bar{g}} = \bar{\bar{I}} \, \delta(\bar{R}/\bar{R}') \tag{27}$$

In (27), $\bar{\bar{I}}$ denotes the unit dyadic, and $\delta(\bar{R}/\bar{R}')$, the three dimensional delta function. The expression of $\bar{\bar{g}}$ for positive values of a is given in reference (5). The derivation is outlined in the Appendix. The expression of $\bar{\bar{g}}$ for negative values of a is also included. We consider now the vector function

$$\bar{A} = \bar{F}_1 \times \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}} \cdot \bar{b}) \right]$$
$$+ (\bar{\bar{\alpha}}^{-1} \cdot \nabla \times \bar{F}_1) \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}} \cdot \bar{b}) \tag{28}$$

Where $\bar{b}$ denotes an arbitrary constant vector. It can easily be verified that

$$\nabla \cdot \bar{A} = \left[\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \nabla \times \bar{F}_1) \right] \cdot \left[\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}} \cdot \bar{b} \right]$$
$$- \bar{F}_1 \cdot \nabla \times \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}} \cdot \bar{b}) \right] \tag{29}$$

In view of (24) and (27), (29) may be written as

$$\nabla \cdot \bar{A} = \bar{f} \cdot (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}} \cdot \bar{b}) - \bar{F}_1 \cdot \bar{b} \, \delta(\bar{R}/\bar{R}') \tag{30}$$

Applying the divergence theorem to (30) and simplifying the result, one finds

$$\bar{F}_1(\bar{R}) = \iiint \bar{f} \cdot \left[\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}} \right] dv$$
$$+ \oiint \left\{ (\hat{n} \times \bar{F}_1) \cdot \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}}) + K \bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}} \right] ds \right. \tag{31}$$

In the region of integration where there is no current source (31) reduces to

$$\overline{F}_1(\overline{R}') = \iint_S \left\{ (\hat{n} \times \overline{F}_1) \cdot \left[\overline{\overline{\alpha}}^{-1} \cdot \nabla \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{\overline{g}}) + K\, \overline{\overline{\alpha}}^{-1} \cdot \overline{\overline{g}} \right] ds \right. \tag{32}$$

The surface integral evaluated at infinity contributes nothing because of the radiatinn condition. By interchanging the roles of $\overline{R}$ and $\overline{R}'$, and rearranging the terms, we have

$$\overline{\overline{\alpha}} \cdot \overline{F}_1(\overline{R}) =$$

$$\iint_S \left[-\nabla \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{\overline{g}}) + K\overline{\overline{g}} \right] \cdot \hat{n} \times \overline{F}(\overline{R}')\, ds' \tag{33}$$

Since

$$\overline{F}_1(\overline{R}) = \overline{F}(\overline{R})\, e^{-j\omega\Omega z} \tag{34}$$

where

$$\overline{F}(\overline{R}) = \overline{E}(\overline{R}) + \Gamma\, \overline{H}(\overline{R}) \tag{35}$$

the expression for $\overline{F}(\overline{R})$ is therefore given by

$$\overline{\overline{\alpha}} \cdot \overline{F}(\overline{R})\, e^{-j\omega\Omega z}$$

$$\iint_S \left[-\nabla \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{\overline{g}}) + K\overline{\overline{g}} \right] \cdot \hat{n} \times \overline{F}(\overline{R}')\, e^{-j\omega\Omega z'}\, ds' \tag{36}$$

Equation (36) describes, in a compact analytical form, Huygens' Principle for an electromagnetic field in a moving medium. When we separate the part without h, and the part with h, the following two equations are obtained

$$\bar{\bar{\alpha}} \cdot \bar{E}(\bar{R})\, e^{-j\omega\Omega z} =$$

$$\iint_S \left\{ -\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}}) \cdot \left[\hat{n} \times \bar{E}(\bar{R}')\right] - j\omega\mu\, \bar{\bar{g}} \cdot \left[\hat{n} \times \bar{H}(\bar{R}')\right] \right\} e^{-j\omega\Omega z'} ds' \tag{37}$$

and

$$\bar{\bar{\alpha}} \cdot \bar{H}(\bar{R})\, e^{-j\omega\Omega z} =$$

$$\iint_S \left\{ -\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}}) \cdot \left[\hat{n} \times \bar{H}(\bar{R}')\right] + j\omega\epsilon \bar{\bar{g}} \cdot \left[\hat{n} \times \bar{E}(\bar{R}')\right] \right\} e^{-j\omega\Omega z'} ds'$$

Equations (36-37), of course, can be devided individually by considering the wave equation for $\bar{E}$ and $\bar{H}$ separately. In addition to its compactness, the use of a combined field vector eliminates a duplicate effort.

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APPENDIX

Derivation of $\bar{\bar{g}}$

The explicit solution to $\bar{\bar{g}}$ which satisfies the equation

$$\nabla \times \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}})\right] - k^2 \bar{\bar{g}} = \bar{\bar{I}}\delta(\bar{R}/\bar{R}') \tag{A.1}$$

can be found most conveniently by using the operational method originally due to Levine and Schwinger[9]. The first term of (A.1) can be decomposed into two terms, namely,

$$\nabla \times \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}})\right]$$
$$= \frac{1}{\alpha}\left[-(\nabla_g \cdot \nabla)\bar{\bar{g}} + \nabla_g\, \nabla \cdot \bar{\bar{g}}\right] \tag{A.2}$$

where

$$\nabla_g = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{a\,\partial z} = \frac{1}{a}\,\bar{\bar{\alpha}}\cdot\nabla$$

Taking the divergence of (A.1) yields

$$\nabla \cdot \bar{\bar{g}} = -\frac{1}{k^2}\nabla\,\delta(\bar{R}/\bar{R}'), \tag{A.3}$$

hence, (A.1) may be written in the form

$$(\nabla_g \cdot \nabla)g + k^2 a\,\bar{\bar{g}} = -\,a\left[\bar{\bar{I}} + \frac{1}{k^2 a^2}\,\bar{\bar{\alpha}}\cdot\nabla\nabla\right]\delta(\bar{R}/\bar{R}') \tag{A.4}$$

If one relates $\bar{\bar{g}}$ with a scalar function g such that

$$\bar{\bar{g}} = a\left[\bar{\bar{I}} + \frac{1}{k^2 a^2}\,\bar{\bar{\alpha}}\cdot\nabla\nabla\right]g \tag{A.5}$$

then g must satisfy the following equation:

$$(\nabla_g \cdot \nabla)\,g + k^2 a\,g = -\,\delta(\bar{R}/\bar{R}')$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{a}\frac{\partial^2}{\partial z^2} + k^2 a\right)g = -\,\delta(\bar{R}/\bar{R}') \tag{A.6}$$

The solution for g in an open region for positive values of a, corresponding to $v < (\mu\epsilon)^{-1/2}$, is given by

$$g = \frac{a^{\frac{1}{2}}\,e^{-jka^{\frac{1}{2}}R_a}}{4\pi R_a} \tag{A.7}$$

where $$R_a = \left[(x - x')^2 + (y - y')^2 + a(z-z')^2\right] .$$

When a is negative, corresponding to $v > (\mu\epsilon)^{-1/2}$, (A.6) has the same form as the two-dimensional Klein-Gordon equation. The solution for g in this case is given by [10]

$$g = \begin{cases} \dfrac{|a|^{\frac{1}{2}} \cos k|a|^{\frac{1}{2}} R'_a}{4\pi R'_a}, & |a|^{\frac{1}{2}}(z-z') > r \\ 0, & |a|^{\frac{1}{2}}(z-z') < r \end{cases}$$

where

$$r = \left[(x-x')^2 + (y-y')^2\right]^{\frac{1}{2}},$$

and $$R'_a = \left[|a|(z-z')^2 - r^2\right]^{\frac{1}{2}} .$$

The discontinuous behavior of g is a manifest of the Cerenkov phenomenon.

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